**On Baxter Q-operators for Toda Chain**

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**Abstract**

We suggest the procedure of the construction of Baxter $Q$-operators for Toda chain . Apart from the one-paramitric family of $Q$-operators, considered in our recent paper [4] we also give the construction of two basic $Q$-operators and the derivation of the functional relations for these operators. Also we have found the relation of the basic $Q$-operators with Bloch solutions of the quantum linear problem.

# 1 Introduction

Long ago, in his famous papers [1] R.Baxter has introduced the object, which is known now as $Q$-operator. This operator was used initially for the solution of the eigenvalue problem of $XYZ$-spin chain, where usual Bethe ansatz fails. Recently this operator was intensively discussed in the series of papers [2] in the connection with continuous quantum field theory. In [3] it was pointed out the relation of $Q$-operator with quantum Bäklund transformations. In [4] we suggested the construction of the one-parametirc family of $Q$-operators for the most difficult case of isotropic Heisenberg spin chain. (In spite of the obvious simplicity of this model, the original Baxter construction fails here.)

The existence of the one-parametric family of $Q$-operators implies the existence of two basic solutions of Baxter equation, whose linear combinations ( with operator coefficients ) form the one-parametric family.

In the present paper we extend the investigation started in [4] to the periodic Toda chain, the other model with rational $R$-matrix. It turns out that apart from the construction of the one-parametric family of $Q$-operators

(section 2), in the case of Toda chain it is possible to build also two basic $Q$-operators separately (section 3). These basic operators satisfy to the set of the functional wronskian relations (section 5), first established for certain field theoretical model in [2]. On the one hand the wronskian relations imply the linear independence of the basic operators, on the other hand they are the origin for numerous fusion relations for the transfer matrix of the model.

In our approach we construct the basic $Q$-operators as the trace of the monodromy of certain $M_n^{(1,2)}(x)$ operators (section 3). It turns out that these operators also permit us to construct the quantum Bloch functions, the basis of the solutions of the quantum linear problem, which are the eigenvectors of the monodromy matrix (section 6).

The defining relation of the $Q$-operator (Baxter equation) for the models with rational $R$-matrix looks as follows:

$$t(x)Q(x) = a(x)Q(x+i) + b(x)Q(x-i), \tag{1}$$

where $t(x)$ is the corresponding transfer matrix and $a(x)$ and $b(x)$ are the c-number functions which enter into factorization of quantum determinant of $t(x)$. In case of Toda chain the quantum determinant is unity, therefore we can choose the normalization $a(x) = b(x) = 1$, which we shall use below.

# 2 Toda Chain

The periodic Toda Chain is the quantum system described by the Hamiltonian

$$H = \sum_{i=1}^{N} \left( p_i^2/2 + \exp(q_{i+1} - q_i) \right), \tag{2}$$

where the canonical variables $p_i, q_i$ satisfy commutation relations

$$[p_i, q_j] = i\delta_{ij} \tag{3}$$

and periodic boundary conditions

$$\begin{aligned} p_{i+N} &= p_i \\ q_{i+N} &= q_i \end{aligned} \tag{4}$$

Following Sklyanin [5] we introduce Lax operator in 2-dimensional auxiliary space as follows:

$$L_n(x) = \left( \begin{array}{cc} x - p_n & e^{q_n} \\ -e^{-q_n} & 0 \end{array} \right), \tag{5}$$

where x is the spectral parameter. The fundamental commutation relations for Lax operator could be written in $R$-matrix form:

$$R_{12}(x-y)L_n^1(x)L_n^2(y) = L_n^2(y)L_n^1(x)R_{12}(x-y), \tag{6}$$

where indexes $1,2$ indicate different auxiliary spaces and $R$-matrix is given by

$$R_{12}(x) = x + iP_{12}, \tag{7}$$

where $P$-is the operator of permutation of the auxiliary spaces. The same intertwining relation also holds true and for the monodromy matrix corresponding to the $L$-operator (5):

$$T_{ij}(x) = \left( \prod_1^N L_n(x) \right)_{ij}, \tag{8}$$

where the multipliers of the product is ordered from the right to the left.

The $Q(x)$-operator we are going to construct will be given as the trace of the monodromy $\hat{Q}(x)$ appropriate operators $M_n(x)$, which acts in n-th quantum space and its auxiliary space, which we will choose to be the representation space $\Gamma$ of the algebra:

$$[\rho_i, \rho_j^+] = \delta_{ij}, \qquad i,j = 1,2 \tag{9}$$

The operator $\hat{Q}(x)$ will be given by the ordered product:

$$\hat{Q}(x) = \prod_{n=1}^N M_n(x), \tag{10}$$

Further we shall need to consider the product $(L_n(x))_{ij}\, M_n(x)$, which acts in the auxiliary space $\Gamma \times C^2$ ($\Gamma$ - for $M_n(x)$ and $C^2$ - is two-dimensional

auxiliary space for $L_n(x)$). In this space it is convenient to consider a pair of projectors $\Pi^{\pm}_{ij}$:

$$\begin{aligned}\Pi^{+}_{ij} &= (\rho^{+}\rho+1)^{-1}\rho_i\rho^{+}_j = \rho_i\rho^{+}_j(\rho^{+}\rho+1)^{-1},\\ \Pi^{-}_{ij} &= (\rho^{+}\rho+1)^{-1}\epsilon_{il}\rho^{+}_l\epsilon_{jm}\rho_m = \epsilon_{il}\rho^{+}_l\epsilon_{jm}\rho_m(\rho^{+}\rho+1)^{-1},\end{aligned} \tag{11}$$

where

$$\begin{aligned}\rho^{+}\rho &= \rho^{+}_i\rho_i\\ \epsilon_{ij} &= -\epsilon_{ji}, \quad \epsilon_{12}=1.\end{aligned} \tag{12}$$

These projectors formally satisfy the following relations:

$$\begin{aligned}\Pi^{\pm}_{ik}\Pi^{\pm}_{kj} &= \Pi^{\pm}_{ij},\\ \Pi^{+}_{ik}\Pi^{-}_{kj} &= 0,\\ \Pi^{+}_{ij}+\Pi^{-}_{ij} &= \delta_{ij}.\end{aligned} \tag{13}$$

Rigorously speaking the r.h.s. of the first equation (13) in the Fock representation has an extra term, proportional to the projector on the vacuum state, but, as we shall see below, this term is irrelevant in the present discussion.

In order to define $Q$-operator which satisfies Baxter equation we shall exploit Baxter's idea [1], which we reformulate as following: $M_n(x)$-*operator should satisfies the relation*:

$$\Pi^{-}_{ij}\left(L_n(x)\right)_{jl} M_n(x)\Pi^{+}_{lk} = 0. \tag{14}$$

If this condition is fulfilled, then

$$\begin{aligned}\left(L_n(x)\right)_{ij} M_n(x) &= \Pi^{+}_{ik}\left(M_n(x)\right)_{kl} M_n(x)\Pi^{+}_{lj} +\\ \Pi^{-}_{ik}\left(L(x)_n\right)_{kl} M_n(x)\Pi^{-}_{lj} &+ \Pi^{+}_{ik}\left(L_n(x)\right)_{kl} M_n(x)\Pi^{-}_{lj}.\end{aligned} \tag{15}$$

In other words, the condition (14) guaranties that the r.h.s. of (15) in the sense of projectors $\Pi^{\pm}$ has the triangle form and this form will be conserved for products over $n$ due to orthogonality of the projectors.

From (14) we obtain

$$\epsilon_{jm}\rho_m\left(L_n(x)\right)_{jk} M_n(x)\rho_k = 0. \tag{16}$$

To satisfy this equation it is sufficient if

$$M_n(x)\rho_k = \left(L_n^{-1}(x)\right)_{kl} \rho_l A_n(x) \tag{17}$$

or

$$\epsilon_{jm}\rho_m \left(L_n(x)\right)_{jk} M_n(x) = B_n(x)\epsilon_{kl}\rho_l, \tag{18}$$

where $A_n(x)$ and $B_n(x)$ are some operators which we shall find now. Note that the operator $L_n^{-1}(x)$ is given by

$$L_n^{-1}(x) = \left( \begin{array}{cc} 0 & -e^{q_n} \\ e^{-q_n} & x-i-p_n \end{array} \right), \tag{19}$$

The equation (18) could be rewritten in the following form:

$$\left(L_n^{-1}(x+i)\right)_{jk} \rho_k M_n(x) = B_n(x)\rho_j \tag{20}$$

Comparing the equations (17) and (20) we conclude that they both are satisfied provided

$$\begin{array}{rcl} A_n(x) & = & M_n(x-i), \\ B_n(x) & = & M_n(x+i). \end{array} \tag{21}$$

In such a way we obtain the following equation for the $M(x)$-operator:

$$\left(L_n^{-1}(x+i)\right)_{jk} \rho_k M_n(x) = M_n(x+i)\rho_j. \tag{22}$$

If the operator $M_n(x)$ satisfies this equation, the product $L_n(x)M_n(x)$ takes the following form:

$$\begin{array}{c} \left(L_n(x)\right)_{ij} M_n(x) = \rho_i M_n(x-i)\rho_j^+(\rho^+\rho+1)^{-1} \\ +(\rho^+\rho+1)^{-1}\epsilon_{il}\rho_l^+ M_n(x+i)\epsilon_{jm}\rho_m + \Pi_{ik}^+ \left(L_n(x)\right)_{kl} M_n(x)\Pi_{lj}^- \quad . \end{array} \tag{23}$$

We do not detail the last term in (23) because, due to triangle structure of it r.h.s. this term will not enter into the trace of $\hat{Q}(x)$.

Now our task is to solve the equation for $M_n(x)$-operator. The detailed investigation of the equation (22) shows that the usual Fock representation for (9) does not fit for our purpose, therefore we shall use less restrictive holomorphic representation.

Let the operator $\rho_i^+$ be the operator of multiplication by the $\alpha_i$, while the operator $\rho_i$-the operator of differentiation with respect to $\alpha_i$:

$$\begin{aligned}\rho_i^+\psi(\alpha) &= \alpha_i\psi(\alpha), \\ \rho_i\psi(\alpha) &= \frac{\partial}{\partial\alpha}\psi(\alpha).\end{aligned} \tag{24}$$

The operators $\rho_i^+, \rho_i$ are canonically conjugated for the scalar product:

$$(\psi, \phi) = \int \frac{\prod_{i=1,2} d\alpha_i d\bar{\alpha}_i}{(2\pi i)^2} e^{-\alpha\bar{\alpha}}\bar{\psi}(\alpha)\phi(\alpha) \tag{25}$$

The action of an operator in holomorphic representation is defined by its kernel:

$$(K\psi)(\alpha) = \int d^2\mu(\beta)K(\alpha, \bar{\beta})\psi(\beta), \tag{26}$$

where we have denoted

$$d^2\mu(\beta) = \frac{\prod_{i=1,2} d\beta_i d\bar{\beta}_i}{(2\pi i)^2}. \tag{27}$$

Now we are ready to make the following

*Statement* The kernel $M_n(x, \alpha, \bar{\beta})$ of the operator $M_n(x)$ in holomorphic representation has the following form:

$$M_n(x, \alpha, \bar{\beta}) = m_n(x)\frac{(\alpha\bar{\beta})^{2l+ix}}{\Gamma(2l + ix + 1)}, \tag{28}$$

where $l$ is arbitrary parameter and the operator $m_n(x)$ is given by

$$\begin{aligned}m_n(x) &= \exp\left[\pi/2(\rho_1^+\rho_2 e^{q_n} - \rho_2^+\rho_1 e^{-q_n}\right]\left(1 + i\rho_2^+\rho_1 e^{-q_n}\right)^{i(p_n - x) + \rho_1^+\rho_1} \\ &= \left(1 - i\rho_1^+\rho_2 e^{q_n}\right)^{i(p_n - x) + \rho_1^+\rho_1}\exp\left[\pi/2(\rho_1^+\rho_2 e^{q_n} - \rho_2^+\rho_1 e^{-q_n}\right]\end{aligned} \tag{29}$$

In (28) the operator $m_n(x)$ acts on the argument $\alpha$ of the function $(\alpha\bar{\beta})^{2l+ix}$ according to (24). The proof of the *Statement* is straightforward by direct substitution of (28) into equation (22). This calculation give us also the by-product – the meaning of the operator $m_n(x)$. Apparently this operator commutes with the operator

$$\hat{l} = \frac{1}{2}(\rho_1^+\rho_1 + \rho_2^+\rho_2). \tag{30}$$

If we shall fix the subspace of $\Gamma$ corresponding to the definite eigenvalue $l$ of the operator $\hat{l}$ then the operator $m_n(x-i(l+1/2))$ becomes Lax operator of Toda chain with auxiliary space, corresponding to the spin $l$. In particular, the operator (5) corresponds to $l=1/2$. Generally speaking, the $m_n(x-i(l+1/2))$ represents Lax operator of Toda chain in the auxiliary space $\Gamma$. This statement could be proved by intertwining of operator (5) with $m_n(x-i(l+1/2))$.

Now, taking the ordered product of the $M_n(x)$ operators we shall obtain the operator $\hat{Q}(x,l)$ whose kernel is given by

$$\begin{aligned}\hat{Q}(x,l,\alpha,\bar{\beta}) &= \int\prod_{i=1}^{N-1} d^2\mu(\gamma_i) M_N(x,l,\alpha,\bar{\gamma}_{N-1}) M_{N-1}(x,l,\gamma_{N-1},\bar{\gamma}_{N-2}) \\ \cdots &\times M_2(x,l,\gamma_2,\bar{\gamma}_1) M_1(x,l,\gamma_1,\bar{\beta}).\end{aligned} \tag{31}$$

Due to triangle (in the sense of projectors $\Pi^{\pm}$ ) structure of the r.h.s. of (23) we obtain the following rule of multiplication of the monodromy matrix $T(x)$ on operator $\hat{Q}(x)$:

$$\begin{aligned}(T(x))_{ij}\,\hat{Q}(x,l,\alpha,\bar{\beta}) = (x+\tfrac{i}{2})^N \rho_i \hat{Q}(x-i,l,\alpha,\bar{\beta})\rho_j^+ (\rho^+\rho+1)^{-1} \\ (x-\tfrac{i}{2})^N(\rho^+\rho+1)^{-1}\epsilon_{im}\rho_m^+\hat{Q}(x+i,l,\alpha,\bar{\beta})\epsilon_{jk}\rho_k + \Pi^+_{im}(\cdots)_{mk}\Pi^-_{kj},\end{aligned} \tag{32}$$

where we omitted the explicit expression of the last term by obvious reason.

To proceed further we need to remind the definition of trace of an operator in holomorphic representation. If the operator is given by its kernel $F(\alpha,\bar{\beta})$ then, (see e.g. [6])

$$TrF = \int d^2\mu(\alpha) F(\alpha,\bar{\alpha}), \tag{33}$$

where the measure was defined in (27). Now we can perform the trace operation for both sides of (32 )over the holomorphic variables and over $i,j$ indexes, corresponding to the auxiliary 2-dimensional space of $T(x)$. The result is the desired Baxter equation:

$$t(x)Q(x,l) = Q(x-i,l) + Q(x+i,l), \tag{34}$$

where, according to (33)

$$Q(x,l) = \int d^2\mu(\alpha)\hat{Q}(x,l,\alpha,\bar{\alpha}). \tag{35}$$

Note, that the trace of $\hat{Q}$ exists due to the exponential factor in holomorphic measure (27) and has cyclic property, therefore $Q(x,l)$ is invariant under cyclic permutation of the quantum variables. Acting as above we can also consider right multiplication $M_n(x)L_n(x)$ to obtain

$$Q(x,l)t(x) = Q(x-i,l) + Q(x+i,l). \tag{36}$$

We shall not consider here the derivations of the intertwining relations for $\hat{Q}(x,l)$ for different values of $x$ and $l$ and for $\hat{Q}(x,l)$ and $T_{ij}(y)$. This may be done in the same way as in [4] and these relations imply the following commutation relations:

$$\begin{aligned} [Q(x,l), Q(y,m)] &= 0 \\ [t(x), Q(y,l)] &= 0 \end{aligned} \tag{37}$$

In such a way we have constructed the family of solutions of the Baxter equation which are parametrized by the parameter $l$. We can prove that this family may be considered as a linear combinations of two basic solutions with operator coefficients. Here arises the question - is it possible to construct these basic operators separately. The answer is positive and now we shall show how our procedure should be modified in this case.

# 3 Basic $Q$-operators for Toda Chain.

As above, we shall look for the $Q$-operators in the form of the monodromy of appropriate $M_n^{(i)}(x)$-operators, which we now supply with the index $i = 1, 2$ and which act in $n$-th quantum space. The auxiliary space $\Gamma$ now will be the representation space of one Heisenberg algebra, instead of (9):

$$[\rho, \rho^+] = 1. \tag{38}$$

The product $(L_n(x))_{ij} M_n^{(i)}(x)$ is an operator in $n$-th quantum space and in auxiliary space which is tensor product $\Gamma \times C^2$. In this auxiliary space we shall introduce new projectors :

$$\begin{aligned} \Pi^+_{ij} &= \begin{pmatrix} 1 \\ \rho \end{pmatrix} \frac{1}{\rho^+\rho + 1} (1, \rho^+), \\ \Pi^-_{ij} &= \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix} \frac{1}{\rho^+\rho + 2} (-\rho, 1) \end{aligned} \tag{39}$$

The defining equations for the operators $M_n^{(i)}$ ( the analogies of eq. (14) ) are

$$\begin{aligned} \Pi^-_{ik}\left(L_n(x)\right)_{kl} M_n^{(1)}(x)\Pi^+_{lj} &= 0, \\ \Pi^+_{ik}\left(L_n(x)\right)_{kl} M_n^{(2)}(x)\Pi^-_{lj} &= 0. \end{aligned} \tag{40}$$

The solutions of these equations we again will present as the kernels of the corresponding operators in holomorphic representation of the algebra (38):

$$\begin{aligned} M_n^{(1)}(x,\alpha,\bar\beta) &= \exp(-i\bar\beta e^{q_n})\frac{e^{-\pi x/2}}{\Gamma(-i(x-p_n)+1)}\exp(i\alpha e^{-q_n}), \\ M_n^{(2)}(x,\alpha,\bar\beta) &= \exp(-i\alpha e^{-q_n})e^{-\pi x/2}e^{(x-p_n)} \\ &\quad \times\Gamma(-i(x-p_n))\exp(i\bar\beta e^{q_n}). \end{aligned} \tag{41}$$

For the right multiplication by $L_n(x)$ these operators automatically satisfy the following equations:

$$\begin{aligned} \Pi^+_{ik}M_n^{(1)}(x)\left(L_n(x)\right)_{kl}\Pi^-_{lj} &= 0, \\ \Pi^-_{ik}M_n^{(2)}(x)\left(L_n(x)\right)_{kl}\Pi^+_{lj} &= 0. \end{aligned} \tag{42}$$

The full multiplication rules for the operators $M_n^i(x)$ and $L_n(x)$ have the following form for left multiplication:

$$\begin{aligned} \left(L_n(x)\right)_{ij} M_n^{(1)}(x) &= \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i M_n^{(1)}(x-i)\frac{1}{\rho^+\rho+1}(1,\rho^+)_j \\ + & \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i \frac{1}{\rho^+\rho+2}M_n^{(1)}(x+i)(-\rho,1)_j + \Pi^+_{ik}\left(L_n(x)\right)_{kl} M_n^{(1)}(x)\Pi^-_{lj} \\ \left(L_n(x)\right)_{ij} M_n^{(2)}(x) &= \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i \frac{1}{\rho^+\rho+1}M_n^{(2)}(x+i)(1,\rho^+)_j \\ + & \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i M_n^{(2)}(x-i)\frac{1}{\rho^+\rho+2}(-\rho,1)_j + \Pi^-_{ik}\left(L_n(x)\right)_{kl} M_n^{(2)}(x)\Pi^+_{lj} \end{aligned} \tag{43}$$

and for right multiplication:

$$M_n^{(1)}(x)\left(L_n(x)\right)_{ij} = \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i \frac{1}{\rho^+\rho+1}M_n^{(1)}(x-i)(1,\rho^+)_j \tag{44}$$

$$+\left(\begin{array}{c}-\rho^+\\1\end{array}\right)_i M_n^{(1)}(x+i)\frac{1}{\rho^+\rho+2}(-\rho,1)_j+\Pi^-_{ik}\left(L_n(x)\right)_{kl}M_n^{(1)}(x)\Pi^+_{lj}$$

$$M_n^{(2)}(x)\left(L_n(x)\right)_{ij}=\left(\begin{array}{c}1\\\rho\end{array}\right)_i M_n^{(2)}(x+i)\frac{1}{\rho^+\rho+1}(1,\rho^+)_j \qquad (45)$$

$$+\left(\begin{array}{c}-\rho^+\\1\end{array}\right)_i \frac{1}{\rho^+\rho+2}M_n^{(2)}(x-i)(-\rho,1)_j+\Pi^+_{ik}\left(L_n(x)\right)_{kl}M_n^{(2)}(x)\Pi^-_{lj}$$

These relations guaranty that the traces of the monodromies, corresponding to both operators $M_n^{(i)}(x)$ satisfy Baxter equations:

$$\begin{aligned} t(x)Q^{(i)}(x) &= Q^{(i)}(x+i)+Q^{(i)}(x-i)\\ Q^{(i)}(x)t(x) &= Q^{(i)}(x+i)+Q^{(i)}(x-i) \end{aligned} \qquad (46)$$

We shall conclude this section with the calculation of the operators $Q^i(x)$ for the simplest case of one quantum degree of freedom. In this case from (33) we easily obtain

$$\begin{aligned} Q^{(1)}(x) &= \int\frac{d\alpha d\bar{\alpha}}{2\pi i}e^{-\alpha\bar{\alpha}}M^1(x,\alpha,\bar{\alpha})=\sum_{n=0}\frac{e^{-\pi x/2}}{n!}e^{-qn}\frac{1}{\Gamma(-i(x-p)+1)}e^{qn}\\ &= \sum_{n=0}\frac{e^{-\pi x/2}}{n!\Gamma(-i(x-p)+n+1)}=e^{-\pi x/2}I_{-i(x-p)}(2), \end{aligned} \qquad (47)$$

where $I_\nu(x)$ is the modified Bessel function. The analogues calculations for the second $Q$-operator gives:

$$\begin{aligned} Q^{(2)}(x) &= -e^{-\pi x/2}\frac{\pi e^{\pi(x-p)}}{\sin\pi i(x-p)}\sum_{n=0}\frac{1}{n!\Gamma(i(x-p)+n+1)}\\ &\quad=-e^{-\pi x/2}\frac{\pi e^{\pi(x-p)}}{\sin\pi i(x-p)}I_{i(x-p)}(2). \end{aligned} \qquad (48)$$

These two expressions could be compared with the results of [7].

# 4 Intertwining Relations.

In this section we shall consider the set of intertwining relations among $L_n(x)$-operator and $M_n^{(i)}(x)$-operators which will imply the mutual commutativity

of transfer matrix and $Q^{(i)}(x)$. Let us start with the simplest relation

$$R^{(i)}_{kl}(x-y)\,(L_n(x))_{lm}\,M^{(i)}_n(y) = M^{(i)}_n(y)\,(L_n(x))_{kl}\,R^{(i)}_{lm}(x-y) \tag{49}$$

From eq. (40) follows that for $x = y$ the $R^{(i)}$-matrixes become the corresponding projectors - $\Pi^-$ for $i = 1$ and $\Pi^+$ for $i = 2$. Making use of these properties we easily obtain:

$$R^{(1)}_{kl}(x-y) = \begin{pmatrix} x-y+i\rho^+\rho & -i\rho^+ \\ -i\rho & i \end{pmatrix} \tag{50}$$

$$R^{(2)}_{kl}(x-y) = \begin{pmatrix} i & i\rho^+ \\ i\rho & x-y+i+i\rho^+\rho \end{pmatrix} \tag{51}$$

Next relation which we shall consider is

$$M^{(1)}_n(x,\rho)M^{(2)}_n(y,\tau)R^{12}(x-y) = R^{12}(x-y)M^{(2)}_n(y,\tau)M^{(1)}_n(x,\rho), \tag{52}$$

where both $M$-operators act in different auxiliary spaces $\Gamma^{(i)}$ and mutual quantum space. The $R$-matrix acts in the tensor product of auxiliary spaces $\Gamma^{(1)} \times \Gamma^{(2)}$. In (52) we have denoted the operators which act in the auxiliary space $\Gamma^{(1)}$ as $\rho, \rho^+$ and operators in $\Gamma^{(2)}$ as $\tau, \tau^+$. From explicit expressions for $M$-operators (41) follows that

$$\begin{aligned} (\rho+\tau)M^{(1)}_n(x,\rho)M^{(2)}_n(y,\tau) &= 0, \\ M^{(2)}_n(y,\tau)M^{(1)}_n(x,\rho)(\rho^++\tau^+) &= 0. \end{aligned} \tag{53}$$

These relations mean that the products of the $M$-operators are triangle operators in the $\Gamma^{(1)} \times \Gamma^{(2)}$ and , as a result the $R$-matrix satisfy the following equations:

$$\begin{gathered} (\rho+\tau)R^{12}(x) = 0 \\ R^{12}(x)(\rho^++\tau^+) = 0. \end{gathered} \tag{54}$$

The corollary of (54) is that the kernel of $R$-matrix in holomorphic representation depends only on one variable:

$$R^{12}(x,\alpha,\bar{\beta};\gamma,\bar{\delta}) = f(x,(\alpha-\gamma)(\bar{\beta}-\bar{\delta})), \tag{55}$$

where the variables $\alpha, \bar{\beta}$ refer to the operators $\rho, \rho^+$ and variables $\gamma, \bar{\delta}$ to the operators $\tau, \tau^+$. Taking (55) into account we can write the intertwining relation (52) in holomorphic representation:

$$\begin{aligned}\int d\mu(\beta')d\mu(\delta')M_n^{(1)}(x,\alpha,\bar{\beta}')M_n^{(2)}(y,\gamma,\bar{\delta}')f(x-y,(\beta'-\delta')(\bar{\beta}-\bar{\delta})) = \\ \int d\mu(\alpha')d\mu(\gamma')f(x-y,(\alpha-\gamma)(\bar{\alpha}'-\bar{\gamma}'))M_n^{(2)}(y,\gamma',\bar{\delta})M_n^{(1)}(x,\alpha',\bar{\beta}),\end{aligned} \tag{56}$$

where

$$d\mu(\alpha) = \frac{d\alpha d\bar{\alpha}}{2\pi i}e^{-\alpha\bar{\alpha}}. \tag{57}$$

To simplify this equation let us introduce the new external variables:

$$\begin{aligned}\xi_1 &= \tfrac{1}{\sqrt{2}}(\alpha+\gamma), & \xi_1' &= \frac{1}{\sqrt{2}}(\beta+\delta), \\ \xi_2 &= \tfrac{1}{\sqrt{2}}(\alpha-\gamma), & \xi_2' &= \frac{1}{\sqrt{2}}(\beta-\delta)\end{aligned} \tag{58}$$

and new integration variables for l.h.s. (r.h.s.) integral:

$$\begin{aligned}\xi_1'' &= \tfrac{1}{\sqrt{2}}(\beta'+\delta') & \left(\xi_1'' = \frac{1}{\sqrt{2}}(\alpha'+\gamma')\right) \\ \xi_2'' &= \tfrac{1}{\sqrt{2}}(\beta'-\delta') & \left(\xi_2'' = \frac{1}{\sqrt{2}}(\alpha'-\gamma')\right).\end{aligned} \tag{59}$$

Apparently, due to the structure of $M^{(i)}$-operators and $R$-matrix, both sides of (56) depend only on the variables $\xi_2, \bar{\xi}_2'$ and integration over $\xi_1''$ becomes trivial, resulting in elimination of these variables in the integrands. Further, representing the function $f(x, 2\xi''\bar{\xi}')$ as

$$f(x, 2\xi''\bar{\xi}') = \sum_{n=0} C_n(x)\frac{(2\xi''\bar{\xi}')^n}{n!}, \tag{60}$$

we can perform the integration over $\xi_2''$ and, comparing similar terms in both sides of (56), conclude that

$$C_n(x) = \frac{1}{\Gamma(-ix+n+1)}. \tag{61}$$

Therefore $R$-matrix in (52) has the following form in holomorphic representation

$$R^{12}(x,\alpha,\bar\beta;\gamma,\bar\delta)=\sum_{n=0}\frac{\Big((\alpha-\gamma)(\bar\beta-\bar\delta)\Big)^n}{n!\Gamma(-ix+n+1)}. \qquad (62)$$

As the operator in the space $\Gamma^{(1)}\times\Gamma^{(2)}$ the $R$-matrix (62) is pathological because its kernel depends only on part of holomorphic variables. In other words it contains the projector $\pi$ on the subspace of $\Gamma^{(1)}\times\Gamma^{(2)}$ which is formed by the functions depending on the difference of variables. This property may be an obstacle in the derivation of the commutativity of $Q$-operators from the intertwining relation (52). The situation is saved due to the same pathological nature of the product of $M$-operators. Indeed, let us consider the product

$$\begin{aligned}Q^{(1)}(x)Q^{(2)}(y)=Tr_1\textstyle\prod_{k=1}^N M_k^{(1)}(x)Tr_2\prod_{k=1}^N M_k^{(2)}(y)=\\ =Tr_{1,2}\textstyle\prod_{k=1}^N M_k^{(1)}(x)M_k^{(2)}(y),\end{aligned} \qquad (63)$$

where the indexes $1,2$ mark the corresponding auxiliary space. Due to the property (53) we can supply each term $M_k^{(1)}(x)M_k^{(2)}(y)$ in the last product with the projector $\pi$. The same holds true also for the product of $Q$-operators taken in the inverse order. In such a way for the commutativity of $Q$-operators we need to consider only the intertwining relations of $M$-operators projected onto the space $\pi\left(\Gamma^{(1)}\times\Gamma^{(2)}\right)$ , where our $R$-matrix is well defined.

Next we shall consider the intertwining relation for the $M^{(1)}$-operators with different values of spectral parameter:

$$R^{(11)}(x-y)M^{(1)}(x,\rho)M^{(1)}(y,\tau)=M^{(1)}(y,\tau)M^{(1)}(x,\rho)R^{(11)}(x-y). \quad (64)$$

As above, the $R$-matrix in (64) acts in the space $\Gamma^{(1)}\times\Gamma^{(2)}$. From explicit expression for $M^{(1)}$-operator (41) we obtain:

$$\rho M^{(1)}(x,\rho)=M^{(1)}(x,\rho)ie^{-q}, \qquad -ie^{q}M^{(1)}(x,\rho)=M^{(1)}(x,\rho)\rho^{+} \qquad (65)$$

These properties of $M^{(1)}$-operator imply the following conditions on the $R$-matrix:

$$\tau^{+}R^{(11)}(x)=R^{(11)}(x)\rho^{+}, \qquad \rho R^{(11)}(x)=R^{(11)}(x)\tau, \qquad (66)$$

which could be satisfied if $R^{(11)}(x)$ has the following form:

$$R^{(11)}(x)=P_{\rho\tau}g(x,\rho^{+}\tau), \qquad (67)$$

where $P_{\rho\tau}$ denotes the operator of permutation of $\rho\tau$ variables. Substituting (67) into relation (64) we get the equation for the function $g$:

$$g(x-y,\rho^+\tau)M^{(1)}(x,\rho)M^{(1)}(y,\tau) = M^{(1)}(y,\tau)M^{(1)}(x,\rho)g(x-y,\rho^+\tau) \quad (68)$$

Making use of the explicit form of the $M^{(1)}$-operator and the formal power series expansion for function $g$ with respect to it second argument we can solve this equation and find the function $g$:

$$g(x,\rho^+\tau) = (1+\rho^+\tau)^{-ix} \quad (69)$$

and therefore

$$R^{(11)}(x) = P_{\rho\tau}(1+\rho^+\tau)^{-ix}. \quad (70)$$

As this $R$-matrix intertwines two similar objects, it should satisfies the Yang-Baxer equation (and it really does), but we shall not investigate further this issue.

The last relation which we need to discuss is the intertwining of two $M^{(2)}$-operators:

$$R^{(22)}(x-y)M^{(2)}(x,\rho)M^{(2)}(y,\tau) = M^{(2)}(y,\tau)M^{(2)}(x,\rho)R^{(22)}(x-y). \quad (71)$$

The $M^{(2)}$-operators also satisfy the relations analogues to (65):

$$\rho M^{(2)}(x,\rho) = -ie^{-q}M^{(2)}(x,\rho), \qquad M^{(2)}(x,\rho)ie^{q} = M^{(2)}(x,\rho)\rho^+, \quad (72)$$

from where we obtain the analogue of (66):

$$\tau^+R^{(22)}(x) = R^{(22)}(x)\rho, \qquad \rho^+R^{(22)}(x) = R^{(22)}(x)\tau^+, \quad (73)$$

and therefore $R^{(22)}$ has the following form:

$$R^{(22)}(x) = P_{\rho\tau}h(x,\tau^+\rho). \quad (74)$$

Further , acting as above we find that the unknown function $h$ does coincide with the function $g$, resulting in the following $R^{(22)}$-matrix:

$$R^{(22)}(x) = P_{\rho\tau}(1+\tau^+\rho)^{-ix}. \quad (75)$$

Now we have completed the derivation of all needed intertwining relation The main corollary of these relations is the mutual commutativity of the transfer matrix and both $Q$-operators:

$$[t(x),Q^{(i)}(y)] = 0, \quad [Q^{(i)}(x),Q^{(j)}(y)] = 0, \quad i(j) = 1,2. \quad (76)$$

## 5 Wronskian-type Functional Relations

It was first pointed out in [2] that Baxter equation (1) which defines the $Q$ -operator could be viewed as the finite difference analogue of the second order differential equation which admits two independent solution. The linear independence of the solutions could be established through the calculation of the wronskian corresponding to the equation. In the previous section we have constructed two solution of Baxter equation and now our task is to prove its linear independence i.e. to derive the finite difference analogue of the wronskian. To solve this problem let us consider in the details the representation of the product (63) of two different $Q$-operators. In the notations of the previous section the product of two $M$-operators which enters into the r.h.s. of (63) has the following form:

$$M_k^{(12)}(x,y,\alpha,\bar{\beta},\gamma,\bar{\delta}) = M_k^{(1)}(x,\alpha,\bar{\beta})M_k^{(2)}(y,\gamma,\bar{\delta}) = \tag{77}$$
$$e^{-\pi(x+y)/2}e^{-i\bar{\beta}e^{q_k}}\frac{1}{\Gamma(-i(x-p_k)+1)}e^{i(\alpha-\gamma)e^{-q_k}}e^{\pi(y-p_k)}\Gamma(-i(y-p_k))e^{i\bar{\delta}e^{q_k}}$$

Changing the holomorphic variables according to (58) we obtain:

$$M_k^{(12)}(x,y,\xi_1,\xi_2,\bar{\xi}_1',\bar{\xi}_2') = e^{-\pi(x+y)/2}e^{-i/\sqrt{2}(\bar{\xi}_1'+\bar{\xi}_2')e^{q_k}}$$
$$\frac{1}{\Gamma(-i(x-p_k)+1)}e^{i\sqrt{2}\xi_2 e^{-q_k}}e^{\pi(y-p_k)}\Gamma(-i(y-p_k))e^{i/\sqrt{2}(\bar{\xi}_1'-\bar{\xi}_2')e^{q_k}} \tag{78}$$

This equation demonstrates that the kernel of $M^{(1)}(x)M^{(2)}(y)$ does not depends on the variable $\xi_1$ and for calculation of the $Q^{(1)}(x)Q^{(2)}(y)$ the dependence of (78) on the variable $\bar{\xi}_1'$ is irrelevant because the integration over $\xi',\bar{\xi}'$ in (63) results in deleting $\bar{\xi}_1'$ from (78). In such a way for the calculation of $Q^{(1)}(x)Q^{(2)}(y)$ we can use instead of $M_k^{(12)}(x,y,\xi_1,\xi_2,\bar{\xi}_1',\bar{\xi}_2')$ the following reduced object:

$$\tilde{M}_k^{(12)}(x,y,\xi,\bar{\xi}') = e^{-\pi(x+y)/2}e^{-i/\sqrt{2}\bar{\xi}'e^{q_k}}$$
$$\frac{1}{\Gamma(-i(x-p_k)+1)}e^{i\sqrt{2}\xi e^{-q_k}}e^{\pi(y-p_k)}\Gamma(-i(y-p_k))e^{-i/\sqrt{2}\bar{\xi}'e^{q_k}}. \tag{79}$$

Note that $\tilde{M}^{(12)}(x,y)$ is nothing else but the kernel of $M^{(1)}(x)M^{(2)}(y)$ on the space $\pi\left(\Gamma^{(1)}\times\Gamma^{(2)}\right)$. Now let use expand the exponents which contain $\xi,\bar{\xi}$

in the r.h.s. of (79) and move the all the factors depending on $p_k$ to the right:

$$\tilde{M}_k^{(12)}(x,y,\xi,\bar{\xi}') = e^{-\pi(x+y)} \sum_{n,m=0} \frac{(i\sqrt{2}\xi)^n}{n!} (-i\bar{\xi}'/\sqrt{2})^m e^{(m-n)q_k}$$
$$\times \sum_{l=0}^{m} \frac{(-1)^{m-l}}{l!(m-l)!} \frac{\Gamma(-i(y-p_k)-m+l)}{\Gamma(-i(x-p_k)+1+n-m+l)} e^{\pi(y-p_k)}. \tag{80}$$

The summation over $l$ in (80) gives:

$$\sum_{l=0}^{m} \frac{(-1)^{m-l}}{l!(m-l)!} \frac{\Gamma(-i(y-p_k)-m+l)}{\Gamma(-i(x-p_k)+1+n-m+l)} =$$
$$\frac{(-1)^m}{m!} \frac{\Gamma(-i(y-p_k)-m)}{\Gamma(-i(x-p_k)+n+1)} \frac{\Gamma(-i(x-y)+m+n+1)}{\Gamma(-i(x-y)+n+1)} \tag{81}$$

and we arrive at the following expression for the $\tilde{M}_k^{(12)}(x,y,\xi,\bar{\xi}')$:

$$\tilde{M}_k^{(12)}(x,y,\xi,\bar{\xi}') = e^{-\pi(x+y)} \sum_{n,m=0} \frac{(i\sqrt{2}\xi)^n}{n!} \frac{(i\bar{\xi}'/\sqrt{2})^m}{m!} e^{(m-n)q_k}$$
$$\times \frac{\Gamma(-i(y-p_k)-m)}{\Gamma(-i(x-p_k)+n+1)} \frac{\Gamma(-i(x-y)+m+n+1)}{\Gamma(-i(x-y)+n+1)} e^{\pi(y-p_k)}. \tag{82}$$

Now let $x$ and $y$ be

$$x = z_+ = z + i(l+1/2), \qquad y = z_- = z - i(l+1/2), \tag{83}$$

where $l$ is an integer (half-integer). For these values of spectral parameters (82) takes the following form:

$$\tilde{M}_k^{(12)}(z_+,z_-,\xi,\bar{\xi}') = e^{-\pi z} \sum_{n,m=0} \frac{(i\sqrt{2}\xi)^n}{n!} \frac{(i\bar{\xi}'/\sqrt{2})^m}{m!} e^{(m-n)q_k}$$
$$\times \frac{\Gamma(-i(z_- - p_k)-m)}{\Gamma(-i(z_+ - p_k)+n+1)} \frac{\Gamma(2l+m+n+2)}{\Gamma(2l+n+2)} e^{\pi(z_- - p_k)}. \tag{84}$$

Further we need to consider (82) for the opposite shift of spectral parameters

$$x = z_- - i\epsilon, \qquad y = z_+ + i\epsilon. \tag{85}$$

We have introduced infinitesimal $\epsilon$ in (85) to remove an ambiguity which arises in (82) for these $x$ and $y$:

$$\tilde{M}_k^{(12)}(z_-, z_+, \xi, \bar{\xi}') = e^{-\pi z} \sum_{n,m=0} \frac{(i\sqrt{2}\xi)^n}{n!} \frac{(i\bar{\xi}'/\sqrt{2})^m}{m!} e^{(m-n)q_k}$$
$$\frac{\Gamma(-i(z_+ - p_k) - m)}{\Gamma(-i(z_- - p_k) + n + l)} \frac{\Gamma(-2l - 2\epsilon + m + n)}{\Gamma(-2l - 2\epsilon + n)} e^{\pi(z_+ - p_k)}. \tag{86}$$

For $\epsilon \to 0$ the fraction of $\Gamma$-functions in (86) takes the following values:

$$\lim_{\epsilon \to 0} \frac{\Gamma(-2l - 2\epsilon + m + n)}{\Gamma(-2l - 2\epsilon + n)} = \begin{cases} \dfrac{\Gamma(-2l + m + n)}{\Gamma(-2l + n)}, & n, m \geq 2l + 1, \\ (-1)^m \dfrac{(2l - n)!}{(2l - n - m)!}, & 2l \geq n + m \geq 0, \\ \dfrac{\Gamma(n + m - 2l)}{\Gamma(n - 2l)}, & n \geq 2l \geq m \\ 0, & otherwise \end{cases} \tag{87}$$

Apparently, the vanishing of the (87) in the fourth region manifests the triangularity of the operator the $\tilde{M}_k^{(12)}(z_-, z_+)$, therefore for the calculation of the trace of the product over $k$ of these operators we need to consider only the part of (87), corresponding to the first two regions. Thus, the resulting expression for the twice reduced operator has the following form:

$$\stackrel{\approx}{M}{}_k^{(12)} (z_-, z_+, \xi, \bar{\xi}') = A_k(z, l, \xi, \bar{\xi}') + B_k(z, l, \xi, \bar{\xi}'), \tag{88}$$

where $A$ contains the part of the r.h.s. of (86) with the summation over $n, m$ in the region $n, m \geq 2l + 1$, $B$ contains the summation over $n, m$ in the region $2l \geq n + m \geq 0$. In other words, the degrees of $\xi, \bar{\xi}'$ in $A$ and $B$ have no intersection and therefore while the calculation of the product $Q^{(1)}(z_-)Q^{(2)}(z_+)$ these two parts will multiply coherently:

$$Q^{(1)}(z_-)Q^{(2)}(z_+) = \int \prod_{k=1}^N d\mu(\xi_k) \stackrel{\approx}{M}{}_N^{(12)} (z_-, z_+, \xi_1, \bar{\xi}_N)$$
$$\times \stackrel{\approx}{M}{}_{N-1}^{(12)} (z_-, z_+, \xi_N, \bar{\xi}_{N-1}) \cdots \stackrel{\approx}{M}{}_1^{(12)} (z_-, z_+, \xi_2, \bar{\xi}_1')$$

$$= \int \prod_{k=1}^N d\mu(\xi_k) A_N(z, l, \xi_1, \bar{\xi}_N) A_{N-1}(z, l, \xi_N, \bar{\xi}_{N-1}) \cdots A_1(z, l, \xi_2, \bar{\xi}_1)$$

$$+ \int \prod_{k=1}^N d\mu(\xi_k) B_N(z, l, \xi_1, \bar{\xi}_N) B_{N-1}(z, l, \xi_N, \bar{\xi}_{N-1}) \cdots B_1(z, l, \xi_2, \bar{\xi}_1) \tag{89}$$

Let us consider first $A$. For the convenience we will shift the values of $n, m$ by $2l+1$, then

$$A_k(z,l,\xi,\bar{\xi}) = (\xi\bar{\xi}')^{2l+1} e^{-\pi z} \sum_{n,m=0} \frac{(i\sqrt{2}\xi)^n}{n!} \frac{(i\bar{\xi}'/\sqrt{2})^m}{(m+2l+1)!} e^{(m-n)q_k}$$
$$\times \frac{\Gamma(-i(z_- - p_k) - m)}{\Gamma(-i(z_+ - p_k) + n + l)} \frac{\Gamma(2l+m+n+2)}{\Gamma(2l+n+2)} e^{\pi(z_- - p_k)}. \tag{90}$$

Comparing (90) with (84), we see that they differ from each other by the factor $(\xi\bar{\xi}')^{2l+1}$ and shift of the factorial $m!$. This difference may be presented as appropriate transformation of $\tilde{M}_k^{(12)}(z_+, z_-, \xi, \bar{\xi}')$ :

$$A_k(z,l,\xi,\bar{\xi}') = \int d\mu(\zeta) d\mu(\zeta') g_l(\xi,\bar{\zeta}) \tilde{M}_k^{(12)}(z_+, z_-, \zeta, \bar{\zeta}') f_l(\zeta', \bar{\xi}'), \tag{91}$$

where

$$g_l(\xi,\bar{\zeta}) = (\xi)^{2l+1} e^{\xi\bar{\zeta}}, \qquad f_l(\zeta,\bar{\xi}) = (\bar{\xi})^{2l+1} \sum_{n=0} \frac{(\zeta\bar{\xi})^n}{(n+2l+1)!}. \tag{92}$$

These two functions possess the following property:

$$\int d\mu(\xi) f_l(\zeta,\bar{\xi}) g_l(\xi,\bar{\zeta}') = e^{\zeta\bar{\zeta}'} \tag{93}$$

The r.h.s. of (93) is the $\delta$-function in holomorphic representation. But note that

$$\int d\mu(\xi) g_l(\zeta,\bar{\xi}) f_l(\xi,\bar{\zeta}') = \sum_{n=2l+1} \frac{(\zeta\bar{\zeta}')^n}{n!}. \tag{94}$$

Taking into account (93), we immediately obtain:

$$\int \prod_{k=1}^N d\mu(\xi_k) A_N(z,l,\xi_1,\bar{\xi}_N) A_{N-1}(z,l,\xi_N,\bar{\xi}_{N-1}) \cdots A_1(z,l,\xi_2,\bar{\xi}_1)$$
$$= Q^{(1)}(z_+) Q^{(2)}(z_-) \tag{95}$$

Our next step is the consideration of $B$ part of the $M^{(12)}(z_-, z_+)$. First of all we shall remove the $\sqrt{2}$ from its holomorphic arguments, because in the integral (89) these factors will cancelled out. Therefore we need to consider the following expression for $B$:

$$B_k(z,l,\xi,\bar{\xi}') = e^{-\pi z} \sum_{t=0}^{2l} \sum_{m=0}^{t} \frac{\xi^{t-m}}{(t-m)!} \frac{\bar{\xi}'^m}{m!} (-1)^m i^{t+2l+1} e^{(2m-t)q_k}$$
$$\times \frac{(2l+m-t)!}{(2l-t)!} \frac{\Gamma(-i(z-p_k)+l-m+1/2)}{\Gamma(-i(z-p_k)-l+t-m+l/2)} e^{\pi(z-p_k)} \tag{96}$$

We intend to compare this operator with Lax operator $L_k^l(x)$ of Toda chain with auxiliary space corresponding to the spin $l$. As it follows from the results of the 2-nd section, $L_k^l(x)$ could be obtain by the reduction of the operator $m_k(x)$ defined in (29) to the subspace corresponding to spin $l$. In the holomorphic representation the kernel of $L_k^l(x)$ could be easily found using the projection:

$$L_k^l(x,\alpha,\bar\beta) = m_k(x - i(l+1/2))\frac{(\alpha\bar\beta)^{2l}}{\Gamma(2l+1)}. \tag{97}$$

(Note that here we again use two-component variables $\alpha_i, \beta_i, i = 1,2$). In (97) the operator $m_k(x)$ should be understood as the differential operator, acting on the projection kernel $\frac{(\alpha\bar\beta)^{2l}}{\Gamma(2l+1)}$. For the calculation of the r.h.s. of the (97) recall that the operator exponential function in (29) is well defined because

$$[i(p-x) + l_3, \rho_1^+\rho_2 e^q] = [i(p-x) + l_3, \rho_2^+\rho_1 e^{-q}] = 0, \tag{98}$$

therefore we can expand the exponential function into formal series and find the action of each term on the projection kernel:

$$m_k(x - i(l+1/2))\frac{(\alpha\bar\beta)^{2l}}{\Gamma(2l+1)} = \tag{99}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(i(p_k-x)+\rho_1^+\rho_1 - l + \frac{1}{2})}{\Gamma(i(p_k-x)+\rho_1^+\rho_1 - l - n + \frac{1}{2})}\frac{(i\rho_1^+\rho_2 e^{q_k})^n}{n!}\frac{(\alpha_1\bar\beta_2 e^{q_k} - \alpha_2\bar\beta_1 e^{-q_k})^{2l}}{\Gamma(2l+1)}.$$

Apparently, only $2l$ terms in (99) will survive because the differential operator $(\rho_2)^n$ acts on the polynomial. The result has the following form:

$$L_k^l(x,\alpha,\bar\beta) = \sum_{t=0}^{2l}\sum_{m=0}^{t} e^{(2m-t)q_k}\frac{\Gamma(-i(x-p_k) - m + l + 1/2)}{\Gamma(-i(x-p_k) - m + t - l + 1/2)}$$
$$(-1)^m i^{2l+t}\frac{\alpha_1^{2l-t+m}\alpha_2^{t-m}\bar\beta_1^{2l-m}\bar\beta_2^{m}}{(2l-t)!(t-m)!m!} \tag{100}$$

This $L$-operator defines the transfer matrix of Toda chain with auxiliary space, corresponding spin $l$:

$$t^l(x) = \int\prod_{k=1}^{N} d^2\mu(\alpha_k) L_N^l(x,\alpha_1,\bar\alpha_N) L_{N-1}^l(x,\alpha_N,\bar\alpha_{N-1})\cdots L_1^l(x,\alpha_2,\bar\alpha_1) \tag{101}$$

If in this formula we will perform the integration over one pair of the holomorphic variables, corresponding for example $\alpha_1, \bar\beta_1$ in (100), the integrand still will be presented in the factorized form, but with new, reduced kernel of $L$-operator:

$$\tilde L^l_k(x,\alpha,\bar\beta) = \sum_{t=0}^{2l}\sum_{m=0}^{t} e^{(2m-t)q_k}\frac{\Gamma(-i(x-p_k)-m+l+1/2)}{\Gamma(-i(x-p_k)-m+t-l+1/2)}$$
$$(-1)^m i^{2l+t}\frac{\alpha_2^{t-m}\bar\beta_2^m}{(2l-t)!(t-m)!m!}(2l-t+m)! \qquad (102)$$

Comparing (102) with (96) we find that

$$B_k(z,l,\xi,\bar\xi') = \tilde L^l_k(z,\xi,\bar\xi) i e^{-\pi p_k}. \qquad (103)$$

Therefore

$$\int \prod_{k=1}^N d\mu(\xi_k) B_N(z,l,\xi_1,\bar\xi_N) B_{N-1}(z,l,\xi_N,\bar\xi_{N-1})\cdots B_1(z,l,\xi_2,\bar\xi_1)$$
$$= i^N t^l(x) e^{-\pi P}, \qquad (104)$$

where

$$P = \sum_{k=0}^N p_k \qquad (105)$$

is the integral of motion, which commutes with $t^l(x)$. In the derivation of (104) we have moved all the factors $e^{-\pi p_k}$ to the right to form $e^{-\pi P}$. Gathering together (89), (95) and (104) we obtain the following functional relations:

$$Q^{(1)}(z-i(l+\tfrac12))Q^{(2)}(z+i(l+\tfrac12)) - Q^{(1)}(z+i(l+\tfrac12))Q^{(2)}(z-i(l+\tfrac12))$$
$$= i^N t^l(x) e^{-\pi P} \qquad (106)$$

For $l=0$ the transfer matrix turns into 1 and we have the simplest wronskian relation:

$$Q^{(1)}(z-i/2)Q^{(2)}(z+i/2) - Q^{(1)}(z+i/2)Q^{(2)}(z-i/2) = i^N e^{-\pi P} \qquad (107)$$

For the illustration of this identity the reader can use the $Q^{(i)}$-operators for one degree of freedom (47), (48). In this simplest case (107) reduces to the well-known identity for Bessel functions:

$$I_\nu(z)I_{-\nu+1}(z) - I_{-\nu}(z)I_{\nu-1}(z) = -\frac{2\sin(\pi\nu)}{\pi z} \qquad (108)$$

The general case (106) for one degree on freedom is related to Lommel polynomials [8].

The functional relations of the type (106) was first established for certain field theoretical model in [2]. In the recent paper of the author with Yu.Stroganov [9] we have discussed the analogues relation for the eigenvalues of $Q$-operators in the case of isotropic Heisenberg spin chain. Originally, since the Baxter paper[1] the existence of one $Q$-operator was considered as important alternative for Bethe ansatz. The relations (106) show the importance of the second $Q$-operator which together with the first one give rise to the numerous fusion relations (see e.g.[2],[9]).

# 6 Discussion

The approach we have considered in the present paper could be applied also to the other with rational $R$-matrix – the discrete self-trapping (DST) model, considered in [3]. The quantum determinant of Lax operator for this model is not unity and Baxter equation has the following form:

$$t(x)Q(x) = (x - i/2)^N Q(x - i) + Q(x + i). \tag{109}$$

The general properties of the $Q$-operators for DST-model are similar to that of Toda system. The eigenvalues of one $Q$-operator are polynomial in spectrum parameter , while the eigenvalues of the second are meromorphic functions. In the case of Toda system the eigenvalues of $Q^{(1)}(x)$ are entire functions, the eigenvalues of $Q^{(2)}(x)$ are meromorphic. For the DST -model there also exist the functional relations similar to (106).

The most interesting would be the application of the formalism to the case of XXX-spin chain. The situation here is the following. In [4] we have constructed the family of $Q(x, l)$-operators similar to (31). Moreover, from the results of [9] follows that for XXX-spin chain there exist the basic $Q$-operators. Making use of the formalism of the section 3, it is possible to the find the $M_k^{(i)}(x)$-operators for this case, but the trace of monodromies corresponding to $M_k^{(i)}(x)$ diverges. This puzzle deserves further investigation.

Another interesting point we want to discuss is the relation of our $M_k^{(i)}(x)$-operators with quantum linear problem for Lax operator (5). In classical case the linear problem is the main ingredient of inverse scattering method, in the same time for the quantum theory it seems to be unnecessary (see for example

excellent review on the subject [10]). However, let us consider the following problem:

$$\psi_{n+1}(x) = L_n(x)\psi_n(x), \tag{110}$$

where $L_n(x)$ is given in (5) and $\psi_n$ is two component quantum operator. From the multiplication rules (43) we obtain:

$$(L_n(x))_{ij}\, M_n^{(1)}(x) \begin{pmatrix} 1 \\ \rho \end{pmatrix}_j = \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i M_n^{(1)}(x.i.). \tag{111}$$

Now let us define the operator

$$\left(\psi_n^{(1)}\right)_i(x) = Tr\left(\prod_{k=n}^{N} M_k^{(1)}(x) \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i \prod_{k=1}^{n-1} M_k^{(1)}(x-i)\right), \tag{112}$$

where the trace is taken over auxiliary space. Apparently, due to (111) the operator (112) does satisfy the equation (110). For $n = 1$, the solution has the following form:

$$\left(\psi_1^{(1)}\right)_i(x) = Tr\left(\prod_{k=1}^{N} M_k^{(1)}(x) \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i\right) = Q^{(1)}(x) \begin{pmatrix} 1 \\ ie^{q_N} \end{pmatrix}_i, \tag{113}$$

where on the last step we have used the explicit form of $Q^{(1)}(x)$-operator for the calculation of the trace. On the other hand the solution (113) translated to the period $N$ by the monodromy (8), due to (111) is given by

$$\left(\psi_{N+1}^{(1)}\right)_i(x) = Tr\left(\begin{pmatrix} 1 \\ \rho \end{pmatrix}_i \prod_{k=1}^{N} M_k^{(1)}(x-i)\right) = Q^{(1)}(x-i) \begin{pmatrix} 1 \\ ie^{q_N} \end{pmatrix}_i. \tag{114}$$

In other words we obtain

$$T_{ij}(x)\left(\psi_1^{(1)}\right)_j(x) = \frac{Q^{(1)}(x-i)}{Q^{(1)}(x)}\left(\psi_1^{(1)}\right)_i(x). \tag{115}$$

This equation may be understood as quantum analogue of the property of Bloch solutions, which are the eigenvectors of the translation to the period.

Similarly we can consider the second solution. Indeed, from the multiplications rules (43) for the $M_n^{(2)}(x)$ we obtain:

$$(L_n(x))_{ij}\, M_n^{(2)}(x) \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_j = \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i M_n^{(2)}(x-i). \tag{116}$$

Therefore the operator

$$\left(\psi_n^{(2)}\right)_i(x) = Tr\left(\prod_{k=n}^{N} M_k^{(2)}(x)\begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i \prod_{k=1}^{n-1} M_k^{(2)}(x-i)\right), \tag{117}$$

possesses the same properties as (112). The initial value of (117) is given by

$$\left(\psi_1^{(2)}\right)_i(x) = Tr\left(\prod_{k=1}^{N} M_k^{(2)}(x)\begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i\right) = Q^{(2)}(x)\begin{pmatrix} -ie^{q_1} \\ 1 \end{pmatrix}_i, \tag{118}$$

where we again on the last step have used the explicit form of $M^{(2)}(x)$. As above we obtain

$$T_{ij}(x)\left(\psi_1^{(2)}\right)_j(x) = \frac{Q^{(2)}(x-i)}{Q^{(2)}(x)}\left(\psi_1^{(2)}\right)_i(x). \tag{119}$$

In such a way using $M_n^{(i)}(x)$-operators we succeeded in the construction of the operators which may be interpreted as the quantum analogues of the Bloch functions of the corresponding linear problem. In the classical theory of finite-zone "potentials", two Bloch solutions of the linear problem, as the functions of spectral parameter are actually the projections of the Backer-Akhiezer function, which is the single-valued meromorphic functions on an hyper-elliptic surface. In quantum case the Bloch functions (112) and (117) do not possess the branching points (in weak sense) which is the trace of the projection in the classical case, therefore their intimate relation is somehow hidden and it will be very interesting to uncover this relationship.

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